Week 9

Q1 Define a sequence S_n by

$$S_1 = 1 \quad S_2 = 1 \qquad S_n = S_{n-1} + 2S_{n-2} \quad \text{for } n \geq 3$$

Let $T \in \mathcal{L}(\mathbb{R}^2)$ defined by $T(x,y) = (y, 2x+y)$

(a) Show that $T^n(0,1) = (S_n, S_{n+1})$

(b) Find the eigenvalues of T

(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

(d) Use the solution of part (c) to compute $T^n(0,1)$.

Given that the eigenvalues of T are real and distinct

Suppose λ_1, λ_2 are eigenvalues of T s.t. $\lambda_1 < \lambda_2$

Prove that

$$S_n = \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2^n - \lambda_1^n\right) \quad \text{for all +ve integer } n.$$

Sol$^{\underline{n}}$ (a) We use induction. For $n=1$, $T(0,1) = (1, 0+1) = (1,1) = (S_1, S_2)$

$\therefore$ Case $n=1$ is true.

Assume $T^k(0,1) = (S_k, S_{k+1})$ is true for some +ve integer k

$T^{k+1}(0,1) = T(S_k, S_{k+1}) = (S_{k+1}, 2S_k + S_{k+1}) = (S_{k+1}, S_{k+2})$

$\therefore$ Case $n = k+1$ is true. By MI. we are done

(b) Let $\beta = \{(1,0),(0,1)\}$ the standard basis of $\mathbb{R}^2$

$M(T,\beta) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ Solving $\det(M(T,\beta) - xI) = 0$

we have

$$0 = \det(M(T,\beta) - xI) = \begin{vmatrix} -x & 1 \\ 2 & 1-x \end{vmatrix} = (-x)(1-x) - 2 = x^2 - x - 2$$
$$= (x-2)(x+1)$$

$\therefore$ The eigenvalues of T are 2 and -1

(c) Let $\lambda_1 = -1$, $\lambda_2 = 2$. $\lambda_2 > \lambda_1$

Solve $(T - \lambda_i I)v = 0$ for v.

$M(T-\lambda_1 I, \beta) = \begin{bmatrix} -\lambda_1 & 1 \\ 2 & 1-\lambda_1 \end{bmatrix}$ Take $v_1 = (1, -\lambda_1)$. Then

$M((T-\lambda_1 I)v_1, \beta) = \begin{bmatrix} -\lambda_1 & 1 \\ 2 & 1-\lambda_1 \end{bmatrix}\begin{bmatrix} 1 \\ -\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2+2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ v_1 is an eigenvector of T with eigenvalue λ_1

$M(T-\lambda_2 I, \beta) = \begin{bmatrix} -\lambda_2 & 1 \\ 2 & 1-\lambda_2 \end{bmatrix}$ Take $v_2 = (1, -\lambda_2)$. Then

$M((T-\lambda_2 I)v_2, \beta) = \begin{bmatrix} -\lambda_2 & 1 \\ 2 & 1-\lambda_2 \end{bmatrix}\begin{bmatrix} 1 \\ -\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ v_2 is an eigenvector of T with eigenvalue λ_2

Then $\gamma = (v_1, v_2)$ is a basis of $\mathbb{R}^2$ consisting of eigenvector of T. (It is lin. indep. by 5.10, $|\gamma| = \dim \mathbb{R}^2 = 2$)

(d) $M(T,\gamma) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $M(Id, \gamma, \beta) = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ $M(Id, \beta, \gamma) = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$

$M(T^n(0,1), \beta) = M(Id, \gamma, \beta)\, M(T^n, \gamma)\, M(Id, \beta, \gamma)\, M((0,1), \beta)$

$= \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^n \left(\frac{1}{\lambda_2-\lambda_1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{1}{\lambda_2-\lambda_1}\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{1}{\lambda_2-\lambda_1}\begin{bmatrix} \lambda_1^n & \lambda_2^n \\ \lambda_1^{n+1} & \lambda_2^{n+1} \end{bmatrix}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{1}{\lambda_2-\lambda_1}\begin{bmatrix} \lambda_1^n\lambda_2 - \lambda_2^n\lambda_1 & \lambda_2^n - \lambda_1^n \\ \lambda_1^{n+1}\lambda_2 - \lambda_2^{n+1}\lambda_1 & \lambda_2^{n+1} - \lambda_1^{n+1} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_2-\lambda_1}(\lambda_2^n - \lambda_1^n) \\ \frac{1}{\lambda_2-\lambda_1}(\lambda_2^{n+1} - \lambda_1^{n+1}) \end{bmatrix}$

$\therefore S_n = \frac{1}{\lambda_2-\lambda_1}\left[\lambda_2^n - \lambda_1^n\right]$ $\forall$ +ve integer n

27. Let $\|\cdot\|$ be a norm on a real vector space V satisfying the parallelogram law given in Exercise 11. Define

$$\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right].$$

Prove that $\langle\cdot,\cdot\rangle$ defines an inner product on V such that $\|x\|^2 = \langle x, x\rangle$ for all $x \in$ V.

Definition. *Let* V *be a vector space over* F, *where* F *is either* R *or* C. *Regardless of whether* V *is or is not an inner product space, we may still define a norm* $\|\cdot\|$ *as a real-valued function on* V *satisfying the following three conditions for all* $x, y \in$ V *and* $a \in F$:

1. $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = 0$.
2. $\|ax\| = |a| \cdot \|x\|$.
3. $\|x+y\| \leq \|x\| + \|y\|$.

Sol^n Recall the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for } x, y \in V. \qquad (4)$$

Recall the definition of inner product $\forall u, v, w \in V, \lambda \in \mathbb{F}$

(IP1) Additivity in 1^{st} slot

$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$$

(IP2) Homogenity in 1^{st} slot

$$\langle \lambda u, w\rangle = \lambda\langle u, w\rangle$$

(IP3) Conjugate symmetry

$$\overline{\langle u, w\rangle} = \langle w, u\rangle$$

(IP4) Positivity

$$\langle u, u\rangle \geq 0$$

(IP5) Definiteness

If $\langle u, u\rangle = 0$ then $u = 0$.

We check (IP4) and $\|u\| = \sqrt{\langle u,u\rangle}$ first

$$\langle u,u\rangle = \frac{1}{4}\left(\|u+u\|^2 - \|u-u\|^2\right) = \frac{1}{4}\|2u\|^2 \underset{②}{=} \frac{1}{4}\left(|2|\,\|u\|\right)^2 = \|u\|^2 \underset{①}{\geq} 0$$

$\therefore \|u\| = \sqrt{\langle u,u\rangle}$

(IP5) If $\langle u,u\rangle = 0$ then $\|u\| = 0$ By ① $u=0$

(IP3)
$$\begin{aligned}\overline{\langle u,w\rangle} &= \overline{\frac{1}{4}\left(\|u+w\|^2 - \|u-w\|^2\right)}\\ &= \frac{1}{4}\left(\|w+u\|^2 - \|(-1)(w-u)\|^2\right)\\ &= \frac{1}{4}\left(\|w+u\|^2 - |-1|^2\|w-u\|^2\right) \quad \text{By ②}\\ &= \frac{1}{4}\left(\|w+u\|^2 - \|w-u\|^2\right) = \langle w,u\rangle\end{aligned}$$

(IP1) $\langle u+v,w\rangle = \frac{1}{4}\left(\|u+v+w\|^2 - \|u+v-w\|^2\right)$

$$= \frac{1}{4}\Big(\frac{1}{2}\big(2\|u+w\|^2 + \cancel{2\|v\|^2} - \cancel{\|u-v+w\|^2} + 2\|v+w\|^2 + \cancel{2\|u\|^2} - \cancel{\|-u+v+w\|^2}\big) \quad \text{By ④}$$

$$-\frac{1}{2}\big(2\|u-w\|^2 + \cancel{2\|v\|^2} - \cancel{\|u-v-w\|^2} + 2\|v-w\|^2 + \cancel{2\|u\|^2} - \cancel{\|-u+v-w\|^2}\big)\Big)$$

$$= \frac{1}{4}\left(\|u+w\|^2 - \|u-w\|^2 + \|v+w\|^2 - \|v-w\|^2\right)$$

$$= \langle u,w\rangle + \langle v,w\rangle$$

The hardest part is (IP2)

We prove it in several steps

(a) $\langle \lambda u,w\rangle = \lambda\langle u,w\rangle$ for $\lambda = 0, 1, 2, \cdots$

Pf For $\lambda=0$ $\langle 0u,w\rangle = \frac{1}{4}\left(\|w\|^2 - \|-w\|^2\right) = 0 = 0\langle u,w\rangle$

Assume $\langle \lambda u,w\rangle = \lambda\langle u,w\rangle$ for some non negative integer λ

$$\langle(\lambda+1)u,w\rangle = \langle \lambda u+u,w\rangle \overset{(IP1)}{=} \langle \lambda u,w\rangle + \langle u,w\rangle = \lambda\langle u,w\rangle + \langle u,w\rangle = (\lambda+1)\langle u,w\rangle$$

By MI we are done

(b) For any +ve integers m, n $\langle \frac{m}{n}u,w\rangle = \frac{m}{n}\langle u,w\rangle$

Pf We have $n\langle \frac{m}{n}u,w\rangle \overset{\text{part (a)}}{=} \langle n(\frac{m}{n}u),w\rangle = \langle mu,w\rangle \overset{\text{part a}}{=} m\langle u,w\rangle$

Since $n \neq 0$ multiply $\frac{1}{n}$ on both sides we have the result.

(c) $\langle -u,w\rangle = -\langle u,w\rangle$

Pf
$$\begin{aligned}\langle -u,w\rangle &= \langle -u,w\rangle + \langle u,w\rangle - \langle u,w\rangle \overset{IP1}{=} \langle -u+u,w\rangle + \langle u,w\rangle\\ &= \langle 0,w\rangle - \langle u,w\rangle \overset{\text{part (a)}}{=} -\langle u,w\rangle\end{aligned}$$

(d) $\forall$ rational no. q, $\langle qu,w\rangle = q\langle u,w\rangle$ Pf Apply (a), (b), (c)

(e) $|\langle u,v\rangle| \le \|u\|\|v\|$.

Pf $(0 \le) \|u+v\| \overset{③}{\le} \|u\| + \|v\|$ $\therefore$ $\|u+v\|^2 \le (\|u\| + \|v\|)^2 = \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$\|u-v\| + \|v\| \ge \|u\|$ $\|u-v\| + \|u\| = \|v-u\| + \|u\| \ge \|v\|$ $\therefore$ $\|u-v\| \ge |\,\|u\| - \|v\|\,| \ge 0$

$\therefore$ $-\|u-v\|^2 \le -(\|u\| - \|v\|)^2 = -\|u\|^2 + 2\|u\|\|v\| - \|v\|^2$

$\therefore$ $\|u+v\|^2 - \|u-v\|^2 \le 4\|u\|\|v\|$. By replacing v with $-v$ we have $|\|u+v\|^2 - \|u-v\|^2| \le 4\|u\|\|v\|$

Hence $\|u\|\|v\| \ge \frac{1}{4}|\|u+v\|^2 - \|u-v\|^2| = |\langle u,v\rangle|$.

(f) $\forall$ real no. λ, $\langle \lambda u, w\rangle = \lambda\langle u,w\rangle$

Pf Here we use some result from analysis:

(I) Given $\lambda \in \mathbb{R}$, for any $\varepsilon > 0$ there exists rational no. q s.t. $|\lambda - q| < \varepsilon$.

(II) If $\lambda \in \mathbb{R}$ such that for any $\varepsilon > 0$ we have $|\lambda| < \varepsilon$, then $\lambda = 0$.

We want to use (II).

Let $\lambda \in \mathbb{R}$ $\forall \varepsilon > 0$ $\exists q \in \mathbb{Q}$ s.t. $|\lambda - q| < \varepsilon$

$|\langle \lambda u, w\rangle - \lambda\langle u,w\rangle| \overset{②}{=} |\langle \lambda u, w\rangle - \lambda\langle u,w\rangle - \langle qu,w\rangle + q\langle u,w\rangle| \overset{IPI}{=} |\langle(\lambda-q)u,w\rangle - (\lambda-q)\langle u,w\rangle|$

$\le |\langle(\lambda-q)u,w\rangle| + |(\lambda-q)\langle u,w\rangle| \overset{(e)}{\le} \|(\lambda-q)u\|\,\|w\| + |\lambda-q|\,\|u\|\|w\|$

$= 2|\lambda-q|\,\|u\|\|w\| < 2\|u\|\|w\|\varepsilon$

Since ε can be arbitrarily small, $2\|u\|\|w\|\varepsilon$ can also be arbitrarily small.

$\therefore$ $\langle \lambda u, w\rangle - \lambda\langle u,w\rangle = 0$ i.e. $\langle \lambda u, w\rangle = \lambda\langle u,w\rangle$